V. Toledano Laredo - 30/5/08 - Stability conditions and Stokes factors (w/ Bridgeland)

Goal: Joyce's wall-crossing for invts. counting semistable objects in abelian categories $\longleftrightarrow$ Stokes phenomena for irregular connections on $\mathbb{P}^1$

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§1. Joyce's work

$\mathcal{A}$ "nice" abelian category ; today: $\mathcal{A} = \mathrm{mod}(R)$, R f.d. alg / $\mathbb{C}$

Def (Bridgeland) ‖ A stability condition on $\mathcal{A}$ is a homomorphism

$$\begin{array}{ccc} Z: K(\mathcal{A}) & \longrightarrow & \mathbb{C} \\ \cup & & \cup \\ K_{>0}(\mathcal{A}) & \longrightarrow & \mathbb{H} = \{\mathrm{Im}\, z > 0\} \quad \text{st.} \ \ldots\ldots \end{array}$$

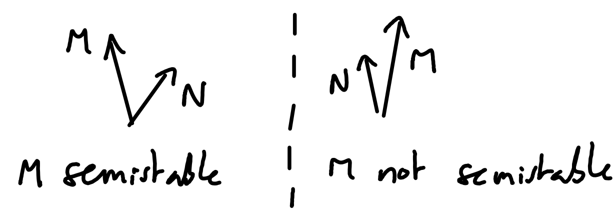
$K_{>0}(\mathcal{A}) \overset{\text{def}}{=} \{[M] / M \in \mathcal{A}\}$ & let $\mathrm{Stab}(\mathcal{A}) := \{Z\}$

★ If R has simple modules $S_1, \ldots, S_N$ then

$$K(\mathcal{A}) \cong \mathbb{Z}[S_1] \oplus \ldots \oplus \mathbb{Z}[S_N], \quad \text{and} \quad \mathrm{Stab}(\mathcal{A}) \cong \mathbb{H}^N$$

Def. ‖
- $0 \neq M \in \mathcal{A} \rightsquigarrow$ • phase of M: $\phi(M) = \frac{1}{\pi} \arg(Z(M)) \in (0,1)$
- M is <u>semistable</u> (wrt Z) if $\forall\, 0 \neq N \subseteq M$, $\phi(N) \leq \phi(M)$

So given $N \subseteq M$, varying Z:

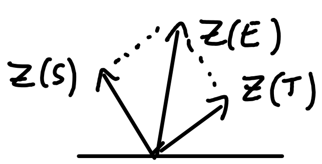


Def: • ‖ for $\alpha \in K_{>0}(\mathcal{A})$, $\Lambda_\alpha = \Lambda_\alpha(Z) := \left\{ M \in \mathcal{A} \,\middle/\, \begin{array}{l} [M] = \alpha \\ M \text{ is } Z\text{-semistable} \end{array} \right\}$

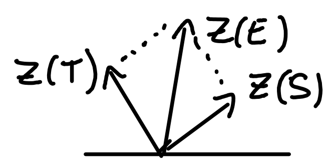
is a projective scheme (A. King)

<u>Wall-crossing behavior</u> ?

<u>Ex</u>: $\mathcal{A} = \mathrm{Rep}\left(\overset{S}{\bullet} \longleftarrow \overset{T}{\bullet} \right)$ • two simples S, T
• $\exists$ nontrivial extension $0 \to S \to E \to T \to 0$



E not semistable
$\Lambda_{[E]} = \emptyset$



E semistable
$\Lambda_{[E]} = \{E\}$

• Jumping locus for $\Lambda_{[E]} := \left\{ Z \,/\, \frac{Z(T)}{Z(S)} \in \mathbb{R}_{>0} \right\}$

A

• Given $\alpha \rightsquigarrow \Lambda_\alpha \rightsquigarrow$ $\delta_\alpha :=$ characteristic f^{n} of Λ_α in the moduli stack $\mathcal{M}$ of all objects of $\mathcal{A}$

$\in$ Hall algebra$(\mathcal{A})$

• $$\mathcal{M} = \coprod_{d \geq 0} \mathcal{M}_d = \coprod_{d \geq 0} \underbrace{\mathrm{Rep}_d(R)}_{d\text{-dim}^{l}\ \text{rep}^{ns}} / GL_d$$

$H_d :=$ GL_d-invariant constructible functions on $\mathrm{Rep}_d(R)$

$$\rightsquigarrow H = \bigoplus_d H_d \ni \delta_\alpha$$

<u>Hall algebra of $\mathcal{A}$</u> with <u>convolution product</u>:

$$f * g(M) := \int_{0 \subseteq N \subseteq M} f(N)\, g(M/N)$$

where $\int$ = Euler characteristic $\int$

i.e. $$\int \sum_i a_i 1_{\mathcal{L}_i} = \sum_i a_i \chi(\mathcal{L}_i)$$

<u>Thm</u>: $H(\mathcal{A})$ is an associative algebra with

- $$f_1 * \dots * f_n(M) = \int_{0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M} f_1(M_1/M_0)\, f_2(M_2/M_1) \dots f_n(M_n/M_{n-1})$$
- unit $= 1_0$ = characteristic function of the zero object 0

& has further structure: let $\begin{cases} \Delta f(M, N) := f(M \oplus N) \\ \eta f := f(0) \end{cases}$

<u>Thm</u> (Ringel, Green, Joyce): $(H, *, 1_0, \Delta, \eta)$ is a bialgebra, co-commutative, with counit η

Primitive elts: $= \{ f \in H \,/\, \Delta f = f \otimes 1_0 + 1_0 \otimes f \}$

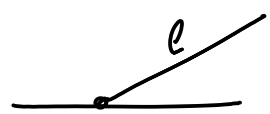
- Enveloping algebra: $\mathcal{U}(\mathcal{A}) = \bigoplus_{\alpha \in K_{>0}(\mathcal{A})} \mathcal{U}(\mathcal{A})_\alpha$ Lie algebra

Ex: $\mathcal{A} = \mathrm{Rep}(Q)$ Q quiver (no loops, no oriented cycles)

$\Rightarrow \mathcal{U}(\mathcal{A}) =$ positive part of the Kac-Moody alg. corresponding to Q

- Next: $\delta_\alpha \in \mathcal{H}_\alpha \rightsquigarrow \varepsilon_\alpha \in \mathcal{U}(\mathcal{A})_\alpha$ as follows

$\ell = \mathbb{R}_{>0}\, e^{i\pi\phi}$ ray



$$\rightsquigarrow \quad SS_\ell := 1 + \sum_{\alpha : Z(\alpha) \in \ell} \delta_\alpha$$

= characteristic function of semistable objects with phase ϕ.

Prop: $\Delta SS_\ell = SS_\ell \otimes SS_\ell$.

let $$\mathrm{Log}\, SS_\ell := \sum_{n \geq 1} (-1)^{n-1} \frac{(SS_\ell - 1)^n}{n} \in \mathcal{U}(\mathcal{A})$$

then $$\mathrm{Log}\, SS_\ell = \sum_{\alpha \,/\, Z(\alpha) \in \ell} \varepsilon_\alpha, \qquad \varepsilon_\alpha \in \mathcal{U}(\mathcal{A})_\alpha$$

Ex: $$\varepsilon_\alpha(M) = \begin{cases} 1 & \text{if } [M] = \alpha \text{ and } M \text{ is stable} \\ \in \mathbb{Q} & \text{if } [M] = \alpha \text{ and } M \text{ semistable} \\ 0 & \text{otherwise.} \end{cases}$$

- $\varepsilon_\alpha : \mathrm{Stab}(\mathcal{A}) \longrightarrow \mathcal{U}(\mathcal{A})_\alpha$ piecewise constant

Joyce: (‡) let $f_\alpha := \sum_{n\geq 1} \sum_{\alpha_1+\dots+\alpha_n=\alpha} J(Z(\alpha_1),\dots,Z(\alpha_n))\, \varepsilon_{\alpha_1} * \dots * \varepsilon_{\alpha_n}$

where $J: (\mathbb{C}^*)^n \to \mathbb{C}$

Thm (Joyce): (1) $\exists \{J_n\}$ s.t. $f_\alpha : \mathrm{Stab}(\mathcal{A}) \to u(\mathcal{A})_\alpha$ as defined above are continuous, holomorphic functions

(2) The J_n are universal and uniquely def^d by (1).

(3) $$df_\alpha = \sum_{\beta+\gamma=\alpha} [f_\beta, f_\gamma]\, d\log(\beta/\gamma) \qquad (*)$$

here view $\beta,\gamma \in K_{>0}(\mathcal{A})$ as coordinates: $\mathrm{Stab}(\mathcal{A}) \to \mathbb{C}$

Observation (Bridgeland + T.)

Suppose
- α ranges over the roots of a semisimple Lie alg. $\mathfrak{g}$
- f_α takes values in $\mathfrak{g}_\alpha$
- $Z \in \mathfrak{h}_{reg}$

Then the PDE $(*)$ is the equation for irregular $\mathfrak{g}$-connections on $\mathbb{P}^1$ of the form $d - \left(\frac{Z}{t^2} + \frac{f}{t}\right) dt$, t coord. on $\mathbb{P}^1$.

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§ Stokes factors:

G linear group $/\mathbb{C}$, $\mathfrak{g} = \mathrm{Lie}\, G$
$\cup$
H torus, $\mathfrak{h} = \mathrm{Lie}\, H$

$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$, $\alpha \in \mathfrak{h}^* \setminus 0$ roots.

Ex: 1) G semisimple or reductive group (e.g. GL_n),
$H =$ max torus

2) $G = H \ltimes N$, H torus, N nilpotent

(Hall algebra case !!)

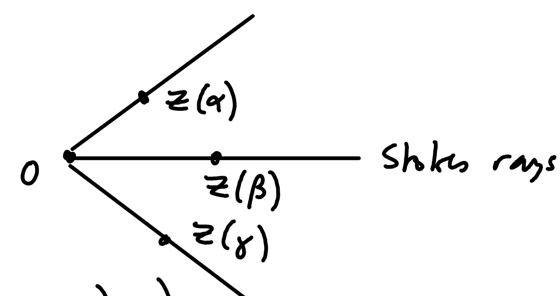
P trivial G-bundle on $\mathbb{P}^1$

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t}\right) dt, \qquad Z \in \mathfrak{h}_{reg}, \quad f \in \bigoplus_\alpha \mathfrak{g}_\alpha$$

Thm: $\exists!$ formal fundamental solution ϕ of ∇ of the form
$\phi = F e^{-Z/t}$, $F \in G[[t]]$, $F(0) = 1$.

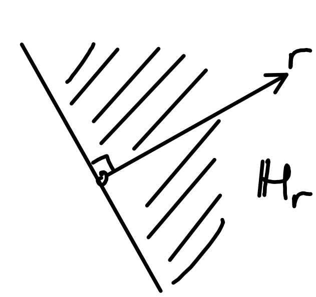
Problem: radius of convergence of F is zero.

Def^n:
- The Stokes rays of ∇ are the rays $\mathbb{R}_+ Z(\alpha)$, α root
- The Stokes sectors are the connected components of $\mathbb{C}^*$ cut out by the rays

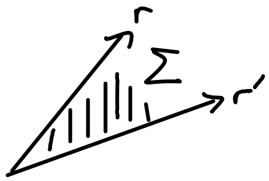


(when $Z = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \in \mathfrak{h}_{reg}$,
the rays are in the direction of $(z_i - z_j)_{i,j}$)

- Choose a ray r which is not a Stokes ray;
$\mathbb{H}_r =$ halfplane in direction of r



Then: $\exists!$ fund solution ϕ of ∇ of the form $\phi = F e^{-Z/t}$
where F is holomorphic in $\mathbb{H}_r$, s.t. $\phi e^{Z/t} \to 1$ as $t \to 0$ in $\mathbb{H}_r$

• 

Given 2 rays r, r' (non-Stokes), the fund. solutions ϕ_r and $\phi_{r'}$ differ by a correction factor $S_\Sigma \in G$: $\phi_r = \phi_{r'} S_\Sigma$

<u>Prop:</u> S_Σ is unipotent and $\log S_\Sigma \in \bigoplus_{\alpha / Z(\alpha) \in \Sigma} \mathfrak{g}_\alpha$

<u>Corollary:</u>
- if Σ doesn't contain any Stokes rays $\Rightarrow S_\Sigma = 1$
- if $\Sigma \ni$ only one Stokes ray ℓ, then
$$\underbrace{S_\ell}_{\text{Stokes factors}} := S_\Sigma = \exp\Big(\sum_{\alpha : Z(\alpha) \in \ell} \varepsilon_\alpha \Big)$$

"Monodromy of ∇" $= \{ S_\ell \}$

$\mathcal{S}$ = <u>Stokes map</u> : $\mathfrak{g} \ominus \mathfrak{g}_0 := \bigoplus_\alpha \mathfrak{g}_\alpha \longrightarrow \mathfrak{g} \ominus \mathfrak{g}_0$

$$f \longmapsto \sum_\alpha \varepsilon_\alpha$$

(residue at ∞ in conn: $\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t} \right) dt$)

<u>Thm</u>: (B-T.) : (1) $\mathcal{S}$ is given by a Lie series of the form

$$\varepsilon_\alpha = \sum_{n \geq 1} \sum_{\alpha_1 + \dots + \alpha_n = \alpha} L_n(Z(\alpha_1) \dots Z(\alpha_n)) f_{\alpha_1} \dots f_{\alpha_n}$$

where $L_1 \equiv 1$

$$L_n(z_1 \dots z_n) = 2\pi i \int_{[0, s_n]} \frac{dt}{t - s_1} \circ \dots \circ \frac{dt}{t - s_n}$$

$s_i = z_1 + \dots + z_i$

(2) The Taylor series of $\mathcal{S}^{-1}$ at $\varepsilon = 0$ is given by a Lie series of the form

$$f_\alpha = \sum_{n\geq 1} \sum_{\alpha_1+\dots+\alpha_n=\alpha} J_n\big(Z(\alpha_1)\dots Z(\alpha_n)\big)\, \varepsilon_{\alpha_1}\dots\varepsilon_{\alpha_n}$$

where the J_n's $\equiv$ Joyce's functions

Application to Joyce's work:

$G = H \ltimes N$ where

$$\begin{cases} H = \mathrm{Hom}(K(\mathcal{A}), \mathbb{C}^*) \\ \mathfrak{h} = \mathrm{Hom}(K(\mathcal{A}), \mathbb{C}) \supset \mathrm{Stab}(\mathcal{A}) \ni Z \\ N = \text{pronilpotent group with Lie alg. } \mathrm{Lie}(N) = \mathfrak{n}(\mathcal{A}) \end{cases}$$

$$\nabla_{\mathcal{A},Z} := d - \left(\frac{Z}{t^2} + \frac{f}{t}\right) dt \quad \text{where } f \in \mathfrak{n}(\mathcal{A}) = \bigoplus_\alpha \mathfrak{n}(\mathcal{A})_\alpha$$

Thm (B-T):

The following are equivalent:

1) The f in $\nabla_{\mathcal{A},Z}$ is given by Joyce's formula $(\ddagger)$

2) $\nabla_{\mathcal{A},Z}$ is the unique connection with Stokes factors
$S_\ell = SS_\ell$ (char. function of all semist. objects of phase $\in \ell$)

3) $\nabla_{\mathcal{A},Z}$ is the unique connection with Stokes multipliers $S_\pm$ given by $S_+ = 1_{\mathcal{A}}$ = Characteristic f^n of all objects ($1_{\mathcal{A}}(M) = 1 \ \forall M$)

$S_- = 1_0$ = char. f^n of zero object

(NB: $S_+ := \widetilde{\prod} S_\ell$ clockwise product of all rays in $\mathbb{H}$)